AP® Calculus BC Exam

SECTION I: Multiple-Choice Questions

DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO.

At a Glance

Total Time
1 hour and 45 minutes
Number of Questions
45
Percent of Total Grade
50%
Writing Instrument
Pencil required

Instructions

Section I of this examination contains 45 multiple-choice questions. Fill in only the ovals for numbers 1 through 45 on your answer sheet.

CALCULATORS MAY NOT BE USED IN THIS PART OF THE EXAMINATION.

Indicate all of your answers to the multiple-choice questions on the answer sheet. No credit will be given for anything written in this exam booklet, but you may use the booklet for notes or scratch work. After you have decided which of the suggested answers is best, completely fill in the corresponding oval on the answer sheet. Give only one answer to each question. If you change an answer, be sure that the previous mark is erased completely. Here is a sample question and answer.

Sample Question

Chicago is a

(A) state
(B) city
(C) country
(D) continent
(E) village

Sample Answer

Ⓐ ● Ⓒ Ⓓ Ⓔ

Use your time effectively, working as quickly as you can without losing accuracy. Do not spend too much time on any one question. Go on to other questions and come back to the ones you have not answered if you have time. It is not expected that everyone will know the answers to all the multiple-choice questions.

About Guessing

Many candidates wonder whether or not to guess the answers to questions about which they are not certain. Multiple choice scores are based on the number of questions answered correctly. Points are not deducted for incorrect answers, and no points are awarded for unanswered questions. Because points are not deducted for incorrect answers, you are encouraged to answer all multiple-choice questions. On any questions you do not know the answer to, you should eliminate as many choices as you can, and then select the best answer among the remaining choices.

CALCULUS BC

SECTION I, Part A

Time—55 Minutes

Number of questions—28

A CALCULATOR MAY NOT BE USED ON THIS PART OF THE EXAMINATION

Directions: Solve each of the following problems, using the available space for scratchwork. After examining the form of the choices, decide which is the best of the choices given and fill in the corresponding oval on the answer sheet. No credit will be given for anything written in the test book. Do not spend too much time on any one problem.

In this test: Unless otherwise specified, the domain of a function *f* is assumed to be the set of all real numbers *x* for which *f*(*x*) is a real number.

1. Which of the following is a *y*-coordinate for the equation $y = \frac{1}{2}x^4 + \frac{2}{3}x^3 - 2x^2 + 6$ when the tangents to the curve equal zero?

(A) $-\frac{35}{6}$

(B) –6

(C) 0

(D) $\frac{34}{3}$

(E) 36

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2. What is the sum of the series $\sqrt{5}-\frac{5}{2}+\frac{5\sqrt{5}}{3}-\frac{25}{4}+\ldots+(-1)^n\frac{\sqrt{5}^{n+1}}{n+1}+\ldots$?

(A) $\ln(1 + \sqrt{5})$

(B) $e_{\sqrt{5}}$

(C) $\ln(\sqrt{5})$

(D) $\sqrt{5}$

(E) The series diverges.

---

3. $\lim_{x\to 0}\frac{\sqrt{x+2}+2x-4}{x^3}$

(A) 0

(B) $\frac{3\sqrt{2}}{64}$

(C) $\frac{\sqrt{2}}{24}$

(D) $\frac{\sqrt{2}}{18}$

(E) Undefined

---

4. Find $\frac{d^2y}{dx^2}$ at $x = 1$ for $y^2 - y = 2x^3 - 3x^2 - 4x + 7$.

(A) $-\frac{26}{9}$

(B) $-\frac{22}{27}$

(C) $-\frac{22}{25}$

(D) $-\frac{10}{9}$

(E) $\frac{26}{9}$

---

5. $\lim_{h\to 0}\frac{(2x^2+4xh+2h^2)-2x^2}{h}$

(A) $2x^2$

(B) $-2x^2$

(C) $4x$

(D) 4

(E) Undefined

---

6. $\int\frac{dx}{4x^2-20x+26}=$

(A) $\tan^{-1}(2x-5)+C$

(B) $\sin^{-1}(x-5)+C$

(C) $\tan^{-1}(x-5)+C$

(D) $\frac{1}{2}\tan^{-1}(2x-5)+C$

(E) $\frac{1}{2}\sin^{-1}(2x-5)+C$

---

7. $\int\frac{14x-12}{(x^2+9)(x+3)}dx=$

(A) $\left(\frac{3}{2}\right)\ln|x^2+9|+\left(\frac{5}{3}\right)\tan^{-1}\frac{x}{3}+3\ln|x+3|+C$

(B) $\left(\frac{3}{2}\right)\ln|x^2+9|+\left(\frac{5}{3}\right)\tan^{-1}\frac{x}{3}-3\ln|x+3|+C$

(C) $\left(\frac{3}{2}\right)\ln|x^2+9|-\left(\frac{5}{3}\right)\tan^{-1}\frac{x}{3}-3\ln|x+3|+C$

(D) $\left(\frac{3}{2}\right)\ln|x^2+9|-\left(\frac{5}{3}\right)\tan^{-1}\frac{x}{3}+3\ln|x+3|+C$

(E) $\left(\frac{3}{2}\right)\ln|x^2+9|+\left(\frac{5}{3}\right)\tan^{-1}\frac{x}{3}-3\ln|x+3|+C$

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8. If $\frac{dy}{dx}=2x^3y$ and $y(0)=4$, find an equation for y in terms of x.

(A) $y=e^{2x^4}$

(B) $y=4e^{2x^4}$

(C) $y=4e^{x^4}$

(D) $y=e^{\frac{x^4}{2}}$

(E) $y=4e^{\frac{x^4}{2}}$

9. Find the derivative of $y^3 = (x + 2)^2(2x - 3)^3$

(A) $\frac{y}{3}\left(\frac{2}{x+2}+\frac{3}{2x-3}\right)$

(B) $\frac{y}{3}\left(\frac{2}{x+2}+\frac{6}{2x-3}\right)$

(C) $\frac{3}{y}\left(\frac{2}{x+2}+\frac{3}{2x-3}\right)$

(D) $\frac{3}{y}\left(\frac{2}{x+2}+\frac{6}{2x-3}\right)$

(E) $\frac{y}{3}\left(\frac{2}{x+2}-\frac{6}{2x-3}\right)$

---

10. $\frac{dy}{dx} = (x^3 - 3)y^2$ and $f(2) = \frac{1}{2}$. Find an equation for y in terms of x.

(A) $y=\frac{4}{12x-x^4}$

(B) $y=\frac{4}{x^4-12x}$

(C) $y=\frac{1}{3x-x^4}-\frac{1}{2}$

(D) $y=\frac{1}{x^4-3x}$

(E) $y=\frac{4}{12x-x^4}+2$

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11. Find the derivative of $y = \cos^{-1}(x^2 + 2x)$.

(A) $\frac{-2x-2}{\sqrt{1-\left(x^2+2x\right)^2}}$

(B) $\frac{2x+2}{\sqrt{1-\left(x^2+2x\right)^2}}$

(C) $\frac{-1}{\sqrt{1-(2x+2)^2}}$

(D) $\frac{1}{\sqrt{1-(2x+2)^2}}$

(E) $\frac{-1}{\sqrt{1-\left(x^2+2x\right)^2}}$

---

12. $\lim_{x\to 2}(x^3 - 5x + 3) =$

(A) 1

(B) 3

(C) 8

(D) 10

(E) 18

---

13. $\frac{d}{dx}(\csc x \sec x) =$

(A) $\sec^2 x - \csc^2 x$

(B) $\sec x - \csc x$

(C) $\csc^2 x - \sec^2 x$

(D) $\sec^2 x + \csc^2 x$

(E) $\csc x + \sec x$

---

14. $\frac{d}{dx}\left(\tan\left(\frac{x^3}{x+1}\right)\right) =$

(A) $\frac{3x^3+2x^2}{(x+1)^2}\sec^2\left(\frac{x^3}{x+1}\right)$

(B) $\frac{2x^3+3x^2}{(x+1)^2}\sec^2\left(\frac{x^3}{x+1}\right)$

(C) $\frac{2x^3-3x^2}{x+1}\sec^2\left(\frac{x^3}{x+1}\right)$

(D) $\frac{2x^3-3x}{(x+1)^2}\sec^2\left(\frac{x^3}{x+1}\right)$

(E) $\frac{2x^3+3x^2}{x+1}\sec^2\left(\frac{x^3}{x+1}\right)$

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15. $\lim_{x\to\infty}\frac{2x^3+4x^2-6x+7}{12x^3+2x^2+4x-9} =$

(A) 0

(B) $\frac{1}{6}$

(C) $\frac{1}{3}$

(D) $\frac{1}{2}$

(E) The limit is undefined.

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16. Where is the tangent line perpendicular to the y-axis for the curve $y = 2x^4 - 4x^2 + 7$ located?

(A) $y = 5$

(B) $y = -7$

(C) $x = 5$

(D) $y = 1$

(E) $x = 7$

---

17. If f is continuous on the interval [–3,3] and differentiable everywhere on (–3,3), find $x = c$, where $f(c)$ is the mean value of $f(x) = x^3 - 3x^2 + x - 4$.

(A) –2

(B) –1

(C) 0

(D) 1

(E) 2

---

18. A toy manufacturer has determined the total profit for a month can be determined by the equation $P = -3x^2 + 30x + 150$, where x is the number of thousands of toys sold. How many thousands of toys

should be sold to maximize the profit that month?

(A) 2

(B) 3

(C) 4

(D) 5

(E) 6

---

19. Find the derivative of $f(x) = x^{x^2}$

(A) $x^{(2x^2)}(1 + 2\ln x)$

(B) $e^{x^2 \ln x}(1 + 2\ln x)$

(C) $x^{x^2}(1 + 2x\ln x)$

(D) $x^{x^2}(x^2 + 2x\ln x)$

(E) $e^{x^2 \ln x}(x + 2x\ln x)$

---

20. If $f(x) = x^2\left(\sqrt[3]{x-4}\right)$, $f'(x) =$

(A) $y\left(\frac{2}{x}+\frac{1}{3x-12}\right)$

(B) $\frac{2}{x}+\frac{1}{3x-12}$

(C) $y\left(\frac{2}{x}+\frac{1}{x-4}\right)$

(D) $y\left(\frac{2}{x}-\frac{1}{3(x-4)}\right)$

(E) $\frac{2}{x}-\frac{1}{x-4}$

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21. Find the equation of the line normal to the graph of $y = \frac{3x^2+6}{x+1}$ at (2,6).

(A) $y = -\frac{1}{2}x + 5$

(B) $y = 2x - 7$

(C) $y = \frac{1}{2}x + 5$

(D) $y = -\frac{1}{2}x + 7$

(E) $y = -2x + 2$

---

22. Find the value of C that satisfies the MVTD for $f(x) = 2x^{\frac{3}{2}} + 5x - 2$ on the interval [0,4].

(A) $\frac{16}{9}$

(B) 1

(C) $\frac{4}{3}$

(D) 9

(E) 8

23. If $y = 7^{2x\cos^2 x}$, then $\frac{dy}{dx} =$

(A) $7^{2x\cos^2 x}(2\cos^2 x + 4x\cos x \sin x)$

(B) $7^{2x\cos^2 x}(2\cos^2 x - 4x \cos x \sin x)$

(C) $7^{2x\cos^2 x}(\ln 7)(2\cos^2 x + 4x\cos x \sin x)$

(D) $49^{x\cos^2 x}(\ln 7)(2\cos^2 x + 4x\cos x \sin x)$

(E) $49^{x\cos^2 x}(\ln 7)(2\cos^2 x - 4x\cos x \sin x)$

24. Find the derivative of the inverse of $y = x^3 - 3x + 5$ when $y = 7$.

(A) $\frac{1}{9}$

(B) 0

(C) 9

(D) 144

(E) $\frac{1}{144}$

25. Find the derivative of the inverse of $f(x) = \sin^2(6\pi - x)$ at $f(x) = \frac{1}{2}$ for $0 \le x \le \frac{\pi}{2}$.

(A) -1

(B) $-\frac{\sqrt{2}}{2}$

(C) $\frac{1}{2}$

(D) $\frac{\sqrt{2}}{2}$

(E) 1

26. $\int(3x^3 - 2x^2 + x - 7)dx =$

(A) $12x^4 - 6x^3 + 2x^2 + 7x + C$

(B) $x^2 - x + C$

(C) $12x^4 - 6x^3 + 2x^2 - 7x + C$

(D) $\frac{3}{4}x^4 - \frac{2}{3}x^3 + \frac{x^2}{2} - 7x + C$

(E) $\frac{3}{4}x^4 + \frac{2}{3}x^3 - \frac{x^2}{2} + 7x + C$

27. Find the derivative of $y = \sqrt{\frac{1+x}{3x-1}}\left(\frac{x+2}{x-1}\right)^2$?

(A) $\frac{y}{2}\left(\frac{1}{x+1} - \frac{1}{3x+1}\right) + 2y\left(\frac{1}{x+2} - \frac{1}{x-1}\right)$

(B) $\frac{y}{2}\left(\frac{1}{1+x} - \frac{3}{3x-1}\right) + \frac{1}{x+2} - \frac{1}{x-1}$

(C) $2y\left(\frac{1}{x+1}-\frac{1}{3x+1}\right)+\frac{y}{2}\left(\frac{1}{x+2}-\frac{1}{x-1}\right)$

(D) $\frac{y}{2}\left(\frac{1}{x+1}-\frac{3}{3x-1}\right)+2y\left(\frac{1}{x+2}-\frac{1}{x-1}\right)$

(E) $2y\left(\frac{1}{1+x}-\frac{3}{3x-1}\right)+\frac{y}{2}\left(\frac{1}{x+2}-\frac{1}{x-1}\right)$

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28. Find $\lim_{x\to\frac{\pi}{2}}\frac{3x}{2\cos x}$.

(A) $\frac{3}{2}$

(B) $-\frac{3}{2}$

(C) $\frac{2}{3}$

(D) 3

(E) 0

END OF PART A, SECTION I

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART A ONLY.

DO NOT GO ON TO PART B UNTIL YOU ARE TOLD TO DO SO.

CALCULUS BC

SECTION I, Part B

Time—50 Minutes

Number of questions—17

A GRAPHING CALCULATOR IS REQUIRED FOR SOME QUESTIONS ON THIS PART OF THE EXAMINATION

Directions: Solve each of the following problems, using the available space for scratchwork. After examining the form of the choices, decide which is the best of the choices given and fill in the corresponding oval on the answer sheet. No credit will be given for anything written in the test book. Do not spend too much time on any one problem.

In this test:

1. The **exact** numerical value of the correct answer does not always appear among the choices given. When this happens, select from among the choices the number that best approximates the exact numerical value.

2. Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

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29. $\int x^2 e^{4x^3+7}\,dx =$

(A) $12e^{4x^3+7} + C$

(B) $e^{4x^3+7} + C$

(C) $\frac{1}{12}e^{4x^3+7} + C$

(D) $\frac{1}{2}e^{4x^3+7} + C$

(E) $2e^{4x^3+7} + C$

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30. $\int \frac{x+7}{(2x-3)(x+6)}\,dx =$

(A) $\frac{17}{30}\ln|2x-3| - \frac{1}{15}\ln|x+6| + C$

(B) $\frac{1}{15}\ln|2x-3| - \frac{17}{30}\ln|x+6| + C$

(C) $\frac{17}{30}\ln|2x-3| + \frac{1}{15}\ln|x+6| + C$

(D) $\frac{17}{15}\ln|2x-3| - \frac{1}{15}\ln|x+6| + C$

(E) $\frac{17}{15}\ln|2x-3| - \frac{1}{15}\ln|x+6| + C$

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31. Find b where $y = x^2 - ax + b$ and $y = x^2 + cx$ have a common tangent at (2,1).

(A) $-\frac{3}{2}$

(B) $\frac{3}{2}$

(C) –8

(D) 8

(E) 0

---

32. Car A and car B leave a town at the same time. Car A drives due north at a rate of 60 km/hr and car B goes east at a rate of 80 km/hr. How fast is the distance between them increasing after 2 hours?

(A) 120 km/hr

(B) 160 km/hr

(C) 100 km/hr

(D) 200 km/hr

(E) 70 km/hr

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33. Approximate cos 91°.

(A) $-\frac{179\pi}{180}$

(B) $\frac{179\pi}{180}$

(C) $\frac{\pi}{180}$

(D) $-\frac{\pi}{180}$

(E) 0

---

34. A 30-foot ladder leaning against a wall is pushed up the wall at a rate of 3 ft/sec. How fast is the ladder sliding across the ground towards the wall when it is 18 feet up the wall from the ground?

(A) $-\frac{9}{4}$ ft/sec

(B) $-\frac{4}{9}$ ft/sec

(C) $-\frac{3}{2}$ ft/sec

(D) -4 ft/sec

(E) -3 ft/sec

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35. Approximate the area under the curve $y = x^2 + 2$ from $x = 1$ to $x = 2$ using four left-endpoint rectangles.

(A) 4.333

(B) 3.969

(C) 4.719

(D) 4.344

(E) 4.328

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36. What is the mean value of $f(x) = \frac{\cos x + 1}{2x^2}$ over the interval (–1,1)?

(A) –196

(B) -1

(C) 1

(D) 196

(E) There is no such value.

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37. Find the second derivative of $y = x^5 - 2x^2 + 7$ at $x = 1$.

(A) 0

(B) 1

(C) 12

(D) 16

(E) 20

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38. What is $\frac{dy}{dx}$ if $y = 3^{x^2}$?

(A) $2x \ln 3(3^{x^2})$

(B) $2x \ln 3$

(C) 3^{x^2}

(D) $2x3^{x^2}$

(E) $\ln 3(3^{x^2})$

---

39. $\int_{\ln 2}^{2} \frac{x^3 + x^2 - 2x}{x^2 + x - 2} dx$

(A) $\frac{(\ln 2)^2 - 4}{2}$

(B) $\frac{4 - (\ln 2)^2}{2}$

(C) $\frac{\ln 4 - 4}{2}$

(D) $\frac{4 - \ln 4}{2}$

(E) $2 - (\ln 2)^2$

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40. Find the derivative of the inverse of $y = 2x^2 - 8x + 9$ at $y = 3$.

(A) $\frac{1}{4}$

(B) $\frac{1}{2}$

(C) 1

(D) 3

(E) 4

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41. A frame is bought for a photo that is 144 in^2. The artist would like to have a mat that is 3 in on the top and bottom and 6 in on each side. Find the dimensions of the frame that will minimize its area.

(A) $(12\sqrt{2} + 12)$ in $\times$ $(6\sqrt{2} + 6)$ in

(B) $(12\sqrt{2} + 6)$ in $\times$ $(6\sqrt{2} + 12)$ in

(C) $(12\sqrt{2})$ in $\times$ $(6\sqrt{2})$ in

(D) $(12\sqrt{2} + 6)$ in $\times$ $(6\sqrt{2} + 3)$ in

(E) $(12\sqrt{2} + 3)$ in $\times$ $(6\sqrt{2} + 6)$ in

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42. What is the length of the curve $y = \frac{2}{3}x^{\frac{3}{2}}$ from $x = 1$ to $x = 6$.

(A) $\frac{4}{3}\sqrt{7} - \frac{14}{3}\sqrt{2}$

(B) $\frac{14}{3}\sqrt{7} + \frac{4}{3}\sqrt{2}$

(C) $\frac{14}{3}\sqrt{7} - \frac{4}{3}\sqrt{2}$

(D) $\frac{4}{3}\sqrt{2} - \frac{14}{3}\sqrt{7}$

(E) $\frac{14}{3}\sqrt{2} + \frac{4}{3}\sqrt{7}$

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43. Find the length of the curve defined by $x = \left(\frac{1}{2}\right)t2 = 7$ and $y = \left(\frac{8}{3}\right)(t + 4)^{\frac{3}{2}}$ from $t = 0$ to $t = 8$.

(A) 36

(B) 64

(C) 80

(D) 96

(E) 100

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44. Use Euler's method with $h = 0.2$ to estimate $y = 1$ if $y' = \frac{y^2 - 1}{2}$ and $y(0) = 0$.

(A) 7.690

(B) 12.730

(C) 13.504

(D) 29.069

(E) 90.676

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45. Which of the following series converges?

(A) $\sum_{n=1}^{\infty}\left(\frac{2}{3}\right)^n$

(B) $\sum_{n=1}^{\infty}\frac{1}{n}$

(C) $\sum_{n=1}^{\infty}\frac{1}{n-1}$

(D) $\sum_{n=1}^{\infty}\frac{4^n}{n^2}$

(E) $\sum_{n=1}^{\infty}3^n$

STOP

END OF PART B, SECTION I

IF YOU FINISH BEFORE TIME IS CALLED, YOU MAY CHECK YOUR WORK ON PART B ONLY.

DO NOT GO ON TO SECTION II UNTIL YOU ARE TOLD TO DO SO.

SECTION II

GENERAL INSTRUCTIONS

You may wish to look over the problems before starting to work on them, since it is not expected that everyone will be able to complete all parts of all problems. All problems are given equal weight, but the parts of a particular problem are not necessarily given equal weight.

A GRAPHING CALCULATOR IS REQUIRED FOR SOME PROBLEMS OR PARTS OF PROBLEMS ON THIS SECTION OF THE EXAMINATION.

- You should write all work for each part of each problem in the space provided for that part in the booklet. Be sure to write clearly and legibly. If you make an error, you may save time by crossing it out rather than trying to erase it. Erased or crossed-out work will not be graded.
- Show all your work. You will be graded on the correctness and completeness of your methods as well as your answers. Correct answers without supporting work may not receive credit.
- Justifications require that you give mathematical (noncalculator) reasons and that you clearly identify functions, graphs, tables, or other objects you use.
- You are permitted to use your calculator to solve an equation, find the derivative of a function at a point, or calculate the value of a definite integral. However, you must clearly indicate the setup of your problem, namely the equation, function, or integral you are using. If you use other built-in features or programs, you must show the mathematical steps necessary to produce your results.
- Your work must be expressed in standard mathematical notation rather than calculator syntax. For example, $\int_1^5 x^2\,dx$ may not be written as fnInt (X^2, X, 1, 5).
- Unless otherwise specified, answers (numeric or algebraic) need not be simplified. If your answer is given as a decimal approximation, it should be correct to three places after the decimal point.
- Unless otherwise specified, the domain of a function f is assumed to be the set of all real numbers x for which $f(x)$ is a real number.

SECTION II, PART A
Time—30 minutes
Number of problems—2

A graphing calculator is required for some problems or parts of problems.

During the timed portion for Part A, you may work only on the problems in Part A.

On Part A, you are permitted to use your calculator to solve an equation, find the derivative of a function at a point, or calculate the value of a definite integral. However, you must clearly indicate the setup of your problem, namely the equation, function, or integral you are using. If you use other built-in features or programs, you must show the mathematical steps necessary to produce your results.

1.

x	2	2.2	2.4	2.6	2.8	3
$\frac{dy}{dx}$	6	5	4	2.5	1	0.5

The equation for y is thrice differentiable for $x > 0$ with $y = 3$ at $x = 2$, the second derivative is equal to 2 at $x = 2$, and the third derivative is 4 at $x = 2$. Values of the first derivative are given for select values of x above.

(a) Write an equation for the tangent line of y at $x = 3$. Use this line to approximate y at $x = 3$.

(b) Use a right endpoint Riemann sum with five subintervals of equal length and values from the table to approximate $\int_2^3 \frac{dy}{dx} dx$. Use this approximation to estimate y at $x = 3$. Show your work.

(c) Use Euler's Method, starting at $x = 2$ with five steps of equal size, to approximate y at $x = 3$. Show your work.

(d) Write a third degree Taylor polynomial for y about $x = 2$. Use it to approximate y at $x = 3$.

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2. Let R be the region bound by $y_1 = 2x^3 - 4x^2 - 8$ and $y_2 = 2x^2 + 8x - 8$.

(a) Find the area of R.

(b) Find the volume of the solid generated when R is revolved about the x-axis for $0 \le x \le 4$

(c) Find the volume of the solid generated when R is revolved about the line $x = 2$ when $-1 \le x \le 0$.

SECTION II, PART B

Time—1 hour

Number of problems—4

No calculator is allowed for these problems.

During the timed portion for Part B, you may continue to work on the problems in Part A without the use of any calculator.

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3. Consider the graph of the polar curve $r = 1 + 2\sin\theta$ for $0 \le \theta \le 2\pi$. Let S be the region bound between the inner and outer loops.

   (a) Write an integral expression for the area of S.

   (b) Write expressions for $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ in terms of θ.

   (c) Write an equation in terms of x and y for the line normal to the graph of the polar curve at the point where $\theta = \frac{3\pi}{2}$. Show your work.

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4. For $t \ge 0$, a particle is moving along a curve so that its position at time t is $(x(t), y(t))$. At time $t = 3$, the particle is at position (3,1). It is known that $\frac{dx}{dt} = e^{-2t}(t + 1)^2$ and $\frac{dy}{dt} = \cos^2 t$

   (a) Is the horizontal movement to the left or to the right at time $t = 3$? Find the slope of the particle's path at $t = 3$.

   (b) Find the y-coordinate of the particle's position at time $t = 6$.

   (c) Find the speed and acceleration of the particle at $t = 6$.

   (d) Find the distance traveled by the particle from time $t = 3$ to $t = 6$.

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5. A particle's position in the xy-plane at any time t is given by $x = 3t^3 - 4$ and $y = 2t^5 - 3t^3$. Find:

   (a) The x and y components of the particle's velocity.

   (b) $\frac{dy}{dx}$ at $t = 3$.

   (c) The acceleration of the particle at $t = 3$.

   (d) The time(s) when the particle is changing direction.

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6. Let $y'_1 = \frac{3x^2}{y_1^2}$ and $y'_2 = 2x^3y_2 - xy_2$.

   (a) If $x = 0$ and $y = 6$, find y_1.

   (b) If $x = -2$ and $y = e^2$, find y_2.

   (c) Use Euler's method to approximate y_1 when $x = 3$. Start at $x = 0$ using three steps. Check your answer against the real value of y_1 at $x = 3$. Is this a reasonable approximation?

STOP

END OF EXAM

ANSWER KEY

Section I

1. D
2. A
3. B
4. B
5. C
6. D
7. B
8. E
9. B
10. A
11. A
12. A
13. A
14. B
15. B
16. A
17. A
18. D
19. C
20. A
21. D
22. A
23. E
24. C
25. E
26. D
27. D
28. B
29. C
30. A
31. E
32. C
33. D
34. A
35. B
36. E
37. D
38. A
39. B
40. A
41. A
42. C
43. D
44. C
45. A

EXPLANATIONS

Section I

1. **D** Take the derivative of y and set it equal to zero. $\frac{dy}{dx} = 2x^3 - 2x^2 - 4x$. When the equation is solved, $x = 0$, $x = -1$, and $x = 2$. Then, plug these values into the original equation for y. When this is done, y at $x = 0$ is 6, y at $x = -1$ is $\frac{23}{6}$ and y at $x = 2$ is $\frac{34}{3}$. The only answer choice of these three that is available is $\frac{34}{3}$.

2. **A** The series is the Taylor series expansion of $\ln(1 + x)$, in which $x = \sqrt{5}$.

3. **B** Use L'Hôpital's Rule to differentiate the top and bottom. Repeat when necessary: Since the limit will exist now, do not differentiate anymore; evaluate the limit at $x = 0$. $\lim_{x\to 0}\frac{1}{16}(x+2)^{-\frac{5}{2}} = \frac{\sqrt{2}}{128}$.

4. **B** First, find the corresponding y value for $x = 1$ by plugging $x = 1$ into $y^2 - y = 2x^3 - 3x^2 - 4x + 7$. Thus, $y = -1$ or $y = 2$. Next, use implicit differentiation to find the first derivative with respect to x: $\frac{dy}{dx} = \frac{6x^2 - 6x - 4}{2y - 1}$. Then, take the second derivative, but do not simplify; plug in the x and y-values found above to solve for the second derivative. $\frac{d^2y}{dx^2} = \frac{(2y-1)(12x-6)-(6x^2-6x-4)(2)\frac{dy}{dx}}{(2y-1)^2}$. At (1,–1), $\frac{d^2y}{dx^2} = -\frac{22}{27}$ and at (1, 2), $\frac{d^2y}{dx^2} = \frac{22}{27}$, so the answer is B.

5. **C** This limit is just the definition of the derivative, so rather than deal with the limit, simply take the derivative of the function. In this case, $f(x)$ and $f'(x)$.

6. **D** Although it does not appear so right away, the solution will be an inverse tangent function. Start by completing the square: $\int \frac{dx}{4x^2 - 20x + 26} = \int \frac{dx}{4x^2 - 20x + 25 + 1} = \int \frac{dx}{(2x-5)^2 + 1}$. Use u-substitution to solve from here. Thus, the final solution is $\frac{1}{2}\tan^{-1}(2x - 5) + C$.

7. **B** Use the Method of Partial Fractions to evaluate: $\frac{Ax+B}{x^2+9} + \frac{C}{x+3} = \frac{14x-12}{(x^2+9)(x+3)}$. Then, $A = 3$, $B = 5$, and $C = -3$.

$$\int \frac{3x-2}{(x+2)^2}\,dx = \int \frac{3x}{x^2+9}\,dx + \int \frac{5}{x^2+9}\,dx - \int \frac{3}{x+3}\,dx$$

$$= \left(\frac{3}{2}\right)\ln\left|x^2+9\right| + \left(\frac{5}{3}\right)\tan^{-1}\frac{x}{3} - 3\ln|x+3| + C.$$

8. **E** Separate the variables to solve the differential equation. $\int \frac{dy}{y} = \int 2x^3\,dx$. Then, $\ln|y| = \frac{1}{2}x^4 + C$. $y = Ce^{\frac{x^4}{2}}$. When the initial point, (0, 4), is plugged in, $C = 4$ and the final answer is $y = 4e^{\frac{x^4}{2}}$.

9. **B** Use logarithms and implicit differentiation to find the derivative. The steps are as follows:

$\ln y^3 = \ln[(x + 2)^2(2x - 3)^3]$

$3 \ln y = 2 \ln(x + 2) + 3 \ln(2x - 3)$

$$\frac{3}{y}\frac{dy}{dx} = \frac{2}{x+2} + \frac{6}{2x-3}$$

$$\frac{dy}{dx} = \frac{y}{3}\left(\frac{2}{x+2} + \frac{6}{2x-3}\right)$$

10. **A** Separate the variables to solve the differential equation: $\int \frac{dy}{y^2} = \int (x^3 - 3)dx$. Then, $y = \frac{1}{3x - \frac{x^4}{4}} + C$. Plug in the initial condition $(2, \frac{1}{2})$ to find $C = 0$, then the final equation is $y = \frac{1}{3x - \frac{x^4}{4}} = \frac{4}{12x - x^4}$.

11. **A** Remember the derivative of the inverse cosine function is just negative the derivative of the inverse sine function, so $\frac{d}{dx}\left(\cos^{-1} u\right) = \frac{-1}{\sqrt{1-u^2}}\frac{du}{dx}$. In this case, $\frac{dy}{dx} = \frac{-2x-2}{\sqrt{1-\left(x^2+2x\right)^2}}$.

12. **A** Since the function is continuous over $x = 2$, evaluate the limit by plugging in 2 for x. $\lim_{x\to 2}(x^3 - 5x + 3) = \lim_{x\to 2}(2^3 - 5(2) + 3) = 1$.

13. **A** Use the product rule and remember your derivatives of trig functions. $\frac{d}{dx}(\csc x \sec x) = \csc x(\sec x \tan x) + \sec x(-\csc x \cot x)$. Simplify by separating each of the trig functions into sine and cosine functions:

$$\left(\frac{1}{\sin x}\right)\left(\frac{1}{\cos x}\right)\left(\frac{\sin x}{\cos x}\right) - \left(\frac{1}{\cos x}\right)\left(\frac{1}{\sin x}\right)\left(\frac{\cos x}{\sin x}\right)$$

$$= \left(\frac{1}{\cos^2 x}\right) - \left(\frac{1}{\sin^2 x}\right) = \sec^2 x - \csc^2 x$$

14. **B** Use u-substitution, the quotient rule, and remember your derivatives of trig functions. $\frac{d}{dx}\left(\tan\left(\frac{x^3}{x+1}\right)\right) =$

$$\frac{(x+1)\left(3x^2\right) - x^3}{(x+1)^2}\sec^2\left(\frac{x^3}{x+1}\right) = \frac{2x^3 + 3x^2}{(x+1)^2}\sec^2\left(\frac{x^3}{x+1}\right)$$

15. **B** Use l'Hôpital's Rule.

$$\lim_{x\to\infty}\frac{2x^3 + 4x^2 - 6x + 7}{12x^3 + 2x^2 + 4x - 9} = \lim_{x\to\infty}\frac{6x^2 + 8x - 6}{36x^2 + 4x + 4} = \lim_{x\to\infty}\frac{12x + 8}{72x + 4}$$

$$= \lim_{x\to\infty}\frac{12}{72} = \frac{12}{72} = \frac{1}{6}$$

16. **A** First, notice the question asks for the tangent line that is perpendicular to the y-axis, or parallel to the x-axis. Any line that is parallel to the x-axis will have an equation in the form "$y =$" and the slope of that line will be 0. Eliminate answer choices C and E. Next, take the first derivative of the equation and set it equal to zero: $\frac{dy}{dx} = 8x^3 - 8x = 0$. Solving the resulting equation, the slope will be zero at $x = -1$, $x = 0$, and $x = 1$. To determine the equation of the tangent lines, plug each value of x into the original equation to solve for the corresponding y-values. At $x = -1$ and $x = 1$, $y = 5$ and at $x = 0$, $y = 7$. Thus, the answer is A.

17. **A** Use the MVTD! $f'(c) = 3c^2 - 6c + 1$, $f(3) = -1$, and $f(-3) = -61$. The MVTD states $f'(c) = \frac{f(a) - f(b)}{a - b}$, where a and b are the endpoints of the interval over which $f(x)$ is continuous and differentiable everywhere

and c is the mean value over that interval. For this problem, $3c^2 - 6c + 1 = \frac{-1+61}{3+3}$. When simplified, $c = -1$ or $c = 3$.

18. **D** In order to maximize the profit, the first derivative of P must be found and set equal to zero. When the first derivative is set equal to zero or does not exist, the maximum or minimum of a function exists. The first derivative, $\frac{dP}{dx} = -6x + 30 = 0$, is solved for x. The critical value is $x = 5$. To ensure that $x = 5$ maximizes the profit, the second derivative is found. If the second derivative is negative at $x = 5$, that critical value corresponds to a relative maximum value. The second derivative is $\frac{d^2P}{dx^2} = -6$. Thus, $x = 5$ is at a relative maximum.

19. **C** Recall $a^x = e^{x \ln a}$, and, if $y = e^u$, then $\frac{dy}{dx} = e^u \frac{du}{dx}$. To simplify the calculations, re-write $f(x) = x^{x^2}$ as $f(x) = e^{x^2 \ln x}$. Now, take the derivative of $f(x)$: $f'(x) = e^{x^2 \ln x}(x + 2x \ln x) = x^{x^2}(x + 2x \ln x)$.

20. **A** Rather than using the product rule and chain rule, use logarithmic differentiation. Take the natural log of both sides of the equation and simplify: $\ln y = \ln\left(x^2\left(\sqrt[3]{x-4}\right)\right) = 2 \ln x + \frac{1}{3}\ln(x - 4)$. Next, use implicit differentiation to find the derivative: $\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{1}{3(x-4)}$. Simplify and solve for $\frac{dy}{dx}$: $\frac{dy}{dx} = y\left(\frac{2}{x} + \frac{1}{3(x-4)}\right)$.

21. **D** First, calculate the derivative of y at (2,6):

$$\frac{dy}{dx} = \frac{(x+1)(6x)-(3x^2+6)}{(x+1)^2} = \frac{(2+1)(6 \bullet 2)-(3(2)^2+6)}{(2+1)^2} = 2.$$

Recall that the slope of a normal line is the opposite reciprocal of the slope of a tangent line, so the slope of the normal is $-\frac{1}{2}$. From there, plug these values into the point-slope formula: $y - 6 = -\frac{1}{2}(x - 2)$, thus, $y = -\frac{1}{2}x + 7$.

22. **A** The MVTD states if $y = f(x)$ is continuous on the interval $[a,b]$, and is differentiable everywhere on the interval (a,b), then there is at least one number c between a and b such that: $f'(c) = \frac{f(b)-f(a)}{b-a}$. As $f(x)$ is continuous, determine $f(0)$ and $f(4)$. $F(0) = -2$ and $f(4) = 34$. Thus, when these values are plugged into the formula, $f'(c) = 9$. Finally, to determine c, set this value equal to the derivative of $f(x)$, $3x^{\frac{1}{2}} + 5 = 9$. $c = \frac{16}{9}$.

23. **E** When $y = a^u$, $\frac{dy}{dx} = a^u (\ln a)\frac{du}{dx}$. For this problem, $u = 2x \cos^2 x$ and $\frac{du}{dx} = 2\cos^2 x - 4x \sin x \cos x$. Then, $\frac{dy}{dx} = 7^{2x\cos^2 x}(\ln 7)(2 \cos^2 x - 4x \sin x \cos x) = 49^{x\cos^2 x}(\ln 7)(2 \cos^2 x - 4x \sin x \cos x)$.

24. **C** At $y = 7$, $x = 2$ or $x = -1$. (Remember, if the equation is not easy to solve, try easy values.) Further, the derivative of the inverse of a function is found by $\left.\frac{d}{dx} f^{-1}(x)\right|_{x=c} = \frac{1}{\left[\frac{d}{dy} f(y)\right]_{y=a}}$. Then, take the derivative of y, $\frac{dy}{dx} = 3x^2 - 3$. At $x = 2$, $\frac{dy}{dx} = 9$ and at $x = -1$, $\frac{dy}{dx} = 0$. Finally, the derivative of the inverse is $\frac{1}{\frac{dy}{dx}}$, which for these values of x are $\frac{1}{9}$ and undefined. Since, undefined is not an answer choice, the answer is A.

25. **E** Since the domain is limited, you don't have to work too hard to find x when $f(x) = \frac{1}{2}$. Test a couple values or solve algebraically and you will find $x = \frac{\pi}{4}$. (Make sure you memorize the sines, cosines, and tangents of 0,

$\frac{\pi}{6}, \frac{\pi}{4}, \frac{\pi}{3}, \frac{\pi}{2}$, and π. Know how to use the unit circle, too!) Further, the derivative of the inverse of a function is found by $\frac{d}{dx} f^{-1}(x)\Big|_{x=c} = \frac{1}{\left[\frac{d}{dy} f(y)\right]_{y=a}}$. Then, take the derivative of y, $f'(x) = -2\sin(6\pi - x)\cos(6\pi - x)$. At $x = \frac{\pi}{4}$, $\frac{dy}{dx} = 1$. Finally, the derivative of the inverse is $\frac{1}{\frac{dy}{dx}}$, which equals 1.

26. **D** Use the Power and Addition rules to integrate:

$$\int(3x^3 - 2x^2 + x - 7)dx = 3\left(\frac{1}{4}x^4\right) - 2\left(\frac{1}{3}x^3\right) + \frac{1}{2}x^2 - 7x + C = \frac{3}{4}x^4 - \frac{2}{3}x^3 + \frac{x^2}{2} - 7x + C.$$

27. **D** Since this function is very complicated, it will probably be easiest to differentiate using logarithmic differentiation, instead of the Quotient, Product, and Chain Rules. First, take the natural log of both sides: $\ln y = \ln\left(\sqrt{\frac{1+x}{3x-1}}\left(\frac{x+2}{x-1}\right)^2\right)$. Use logarithmic rules to simplify the equation: $\ln y = \frac{1}{2}[\ln(1 + x) - \ln(3x - 1)] + 2[\ln(x + 2) - \ln(x - 1)]$. Now, differentiate both sides with respect to x: $\frac{1}{y}\frac{dy}{dx} = \frac{1}{2}\left(\frac{1}{x+1} - \frac{3}{3x-1}\right) + 2\left(\frac{1}{x+2} - \frac{1}{x-1}\right)$. Finally, isolate $\frac{dy}{dx}$: $\frac{dy}{dx} = \frac{y}{2}\left(\frac{1}{x+1} - \frac{3}{3x-1}\right) + 2y\left(\frac{1}{x+2} - \frac{1}{x-1}\right)$.

28. **B** When you insert $\frac{\pi}{2}$ for x, the limit is $\frac{3\pi/2}{0}$, which is indeterminate. Use L'Hôpital's Rule to evaluate the limit: $\lim_{x\to\frac{\pi}{2}} \frac{3}{-2\sin x}$. This limit exists and equals $-\frac{3}{2}$.

29. **C** Recall, $\int e^u\,du = e^u + C$. In this problem, $u = 4x^3 + 7$ and $du = 12x^2dx$. Thus, $\frac{1}{12}\int e^u du = \frac{1}{12}e^u + C = \frac{1}{12}e^{4x^3+7} + C$.

30. **A** Use the method of partial fractions to solve this integral. First, split the fraction into parts in equation form $\frac{A}{2x-3} + \frac{B}{x+6} = \frac{x+7}{(2x-3)(x+6)}$. Then, solve for A and B; $A = \frac{17}{15}$ and $B = -\frac{1}{15}$. Now, you can rewrite the fraction as the sum of two fractions and integrate those separately: $\int\frac{x+7}{(2x-3)(x+6)}dx = \frac{17}{15}\int\frac{dx}{2x-3} - \frac{1}{15}\int\frac{dx}{x+6}$. Use u-substitution to integrate both parts and simplify to get the final solution: $\int\frac{x+7}{(2x-3)(x+6)}dx = \frac{17}{30}\ln|2x - 3| - \frac{1}{15}\ln|x + 6| + C$.

31. **E** Take the derivative of both equations and set them equal to each other: $2x - a = 2x + C$, thus $-a = c$. Next, plug in the point, (2,1), into the equations for both curves and simplify. Therefore, $-3 = -2a + b$ and $1 = 4 + 2c$. When these equations are solved $c = -\frac{3}{2}$, $a = \frac{3}{2}$, and $b = 0$.

32. **C** Since the cars are going in two directions that are orthogonal to each other, the distances they travel and the distance between them form a right triangle, in which A is the distance car A travels, B is the distance car B travels, and D is the distance between them: $A^2 + B^2 + D^2$. Because this equation is relating distances, and we were given a point in time, the distances the two cars traveled at that point (2 hours) must be determined. Using the rate formula, distance = rate • time, the speed the cars are traveling multiplied by the time will give us their distances. Car A then traveled 120 km in 2 hours and Car B traveled 160 km. Use these values to determine D: 200 km. Next, differentiate the equation relating their distances with respect to time: $2A\frac{dA}{dt} + 2B\frac{dB}{dt} = 2D\frac{dD}{dt}$. Plug in the values for A, $\frac{dA}{dt}$, B, $\frac{dB}{dt}$, and D into the equation and solve for $\frac{dD}{dt}$: $\frac{dD}{dt} = 100$ km/hr.

33. **D** Use a differential to approximate cos 91°, but be careful! First, you must convert from degrees to radians, so you will be approximating $\cos\frac{91\pi}{180}$ since $1° = \frac{\pi}{180}$. Then, recall the general formula is $f(x + \Delta x) \approx f(x) + f'(x)\Delta x$. For this problem, $f(x) = \cos\frac{\pi}{2} = 0$, $f'(x) = -\sin\frac{\pi}{2} = -1$, and $\Delta x = \frac{\pi}{180}$. When these values are input, the equation is $\cos\frac{91\pi}{180} \approx 0 - 1\left(\frac{\pi}{180}\right) \approx -\frac{\pi}{180}$.

34. **A** The ladder makes a right triangle with the wall and the ground, so the relationship between the three can be found using the Pythagorean theorem, in which we will call x the distance the bottom of the ladder is from the building across the ground and y the distance the top of the ladder is from the ground up the building, so $x^2 + y^2 = 30^2$. Since we want to find the rate that the top of the ladder is sliding, we need to differentiate this equation with respect to t: $2x\frac{dx}{dt} + 2y\frac{dy}{dt} = 0$. We already know $\frac{dy}{dt} = 3$ ft/sec and our y at the time of interest is 18 ft. In order to determine x at that time, plug 18 into $x^2 + y^2 = 30^2$ and solve for x. Thus, $x = 24$ feet. Plug these values into the differentiated equation and solve for $\frac{dx}{dt}$: $2(24)\left(\frac{dx}{dt}\right) + 2(18)(3) = 0$, so $\frac{dy}{dt} = -\frac{9}{4}$ ft/sec.

35. **B** The formula for the area under a curve using left-endpoint rectangles is: $A = \left(\frac{b-a}{n}\right)(y_0 + y_1 + y_2 + y_3 + \ldots + y_n)$, where a and b are the x-values that bound the area and n is the number of rectangles. Since we are interested in the left-endpoints, the x-coordinates are $x_0 = 1$, $x_1 = \frac{5}{4}$, $x_2 = \frac{3}{2}$, and $x_3 = \frac{7}{4}$. The y-coordinates are found by plugging these values into the equation for y, so $y_0 = 3$, $y_1 = 3.5625$, $y_2 = 4.25$, and $y_3 = 5.0625$. Then, $A = \left(\frac{2-1}{4}\right)(3 + 3.5625 + 4.25 + 5.0625) = 3.96875$.

36. **E** Whether you use the MVTD or the MVTI, the requirement is that the function be continuous over the interval in question. $f(x)$ is not continuous over the interval (–1, 1); therefore, there is no mean value.

37. **D** $\frac{dy}{dx} = 5x^4 - 4x$; $\frac{d^2y}{dx^2} = 20x - 4$. Plug in $x = 1$ to evaluate the second derivative, so $\frac{d^2y}{dx^2} = 16$.

38. **A** Recall, $\frac{dy}{dx} = a^u \ln a\left(\frac{du}{dx}\right)$. In this problem, $u = x^2$ and $du = 2x\,dx$. Thus, $\frac{dy}{dx} = 3^{x^2} \ln 3(2x)$.

39. **B** $\int_{\ln 2}^{2} \frac{x^3 + x^2 - 2x}{x^2 + x - 2}\,dx = \int_{\ln 2}^{2} \frac{x\left(x^2 + x - 2\right)}{x^2 + x - 2}\,dx = \int_{\ln 2}^{2} x\,dx = \frac{x^2}{2}\Bigg|_{\ln 2}^{2} = 2 - \frac{(\ln 2)^2}{2}$

40. **A** Recall, the derivative of the inverse is given by $\frac{d}{dx} f^{-1}(x)\big|_{x=c} = \frac{1}{\left[\frac{d}{dy} f(y)\right]_{y=a}}$. First, take the derivative of y, $\frac{dy}{dx} = 4x - 8$. Next, use the given y value, $y = 3$ and solve for their corresponding x values, so $x = 1$ or $x = 3$. Now, following the formula above, solve for the derivative of the inverse using the x values: $\frac{1}{4(1)-8} = -\frac{1}{4}$ and $\frac{1}{4(3)-8} = \frac{1}{4}$. Thus, the answer is A.

41. **A** The area of the photo, mat, and frame is found by the equation $A = (x + 12)(y + 6)$, where x is the width (side-side) of the photo and y the length (top-bottom). If the equation is expanded out, the area is $A = xy + 6x + 12y + 72$, where $xy = 144$ and $y = 144/x$. The final equation, in one variable, is $A = 6x + \frac{1728}{x} + 216$. In order to minimize the area, we take the derivative, set it equal to zero and solve for x: $\frac{dA}{dx} = 6 - \frac{1728}{x^2} = 0$, and $x = 12\sqrt{2}$. To confirm that will minimize the area, take the second derivative of A and

confirm that it is greater than zero at $x = 12\sqrt{2}$: $\frac{d^2A}{dx^2} = \frac{3456}{x^3} > 0$, so $x = 12\sqrt{2}$ will minimize the area of the frame. Use x to solve for y: $y = \frac{144}{12\sqrt{2}} = 6\sqrt{2}$. Recall, x and y are the dimensions of the photo, so the dimensions of the frame will be these values plus the mat, so the dimensions of the frame are $(12\sqrt{2} + 12)$ in $\times (6\sqrt{2} + 6)$in.

42. **C** $L = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$. For this question, $\frac{dy}{dx} = x^{\frac{1}{2}}$from $a = 1$ to $b = 6$. Then, using u-substitution, evaluate: $L = \int_1^6 \sqrt{1+\left(x^{\frac{1}{2}}\right)^2}\, dx = \frac{14}{3}\sqrt{7} - \frac{4}{3}\sqrt{2}$

43. **D** $L = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$. For this question, $\frac{dx}{dt} = t$ and $\frac{dy}{dt} = 4\sqrt{t+4}$ from $a = 0$ to $b = 8$. Then, using u-substitution, evaluate: $L = \int_0^8 \sqrt{t^2 + \left(4(t+4)^{\frac{1}{2}}\right)^2}\, dt = \int_0^8 \sqrt{(t+8)^2}\, dt = 96$

44. **C** Remember, for Euler's Method, $x_n = x_{n-1} + h$ and $y_n = y_{n-1} + h(y_{n-1})$. In this case, $x_0 = 0$, $y_0 = 2$, $y_0' = 5$; $x_1 = 0.2$, $y_1 = 3$, $y_1' = 4$; $x_2 = 0.4$, $y_2 = 3.8$, $y_2' = 6.72$; $x_3 = 0.6$, $y_3 = 5.144$, $y_3' = 12.730$; $x_4 = 0.8$, $y_4 = 7.69$, $y_4' = 29.069$, $x_5 = 1.0$, $y_5 = 13.504$

45. **A** A is a geometric series with $r < 1$, so it converges. B is a harmonic series, so it diverges (verify this with the integral test). C diverges by the comparison test (compare it to the harmonic series). D diverges by the ratio test. Finally, E is a geometric series and diverges because $r > 1$.

Section II

1.

x	2	2.2	2.4	2.6	2.8	3
$\frac{dy}{dx}$	6	5	4	2.5	1	0.5

The equation for y is thrice differentiable for $x > 0$ with $y = 3$ at $x = 2$, the second derivative is equal to 2 at $x = 2$, and the third derivative is 4 at $x = 2$. Values of the first derivative are given for select values of x above.

(a) Write an equation for the tangent line of y at $x = 3$. Use this line to approximate y at $x = 3$.

(a) Since the first derivative of y at $x = 2$ is 6 and y at $x = 2$ is 3, we can write the equation of the tangent line: $y - 3 = 6(x - 2)$. When you insert 3 for x into this equation, $y = 9$.

(b) Use a right endpoint Riemann sum with five subintervals of equal length and values from the table to approximate $\int_2^3 \frac{dy}{dx}\, dx$. Use this approximation to estimate y at $x = 3$. Show your work.

(b) The right endpoint Riemann sum will look like this: $\int_2^3 \frac{dy}{dx}\, dx = (0.2)(5) + (0.2)(4) + (0.2)(4) + (0.2)(2.5) + (0.2)(1) + (0.2)(0.5) = 2.6$. To approximate y at $x = 3$, add this value to y at $x = 2$: $y\,|_{x=3} = y\,|_{x=2} + \int_2^3 \frac{dy}{dx}\, dx$. So, y at $x = 3$ is 5.6.

(c) Use Euler's Method, starting at $x = 2$ with five steps of equal size to approximate y at $x = 3$. Show your work.

(c) Following Euler's Method:

$x_0 = 2, y_0 = 3, y_0' = 6; x_1 = 2.2, y_1 = 4.2, y_1' = 5; x_2 = 2.4, y_2 = 5.2, y_2' = 4; x_3 = 2.6, y_3 = 6, y_3' = 2.5; x_4 = 2.8, y_4 = 6.5, y_4' = 1; x_5 = 3, y_5 = 6.7$

Thus, by Euler's Method, $y = 6.7$ at $x = 3$.

(d) Write a third degree Taylor polynomial for y about $x = 2$. Use it to approximate y at $x = 3$.

(d) A Taylor polynomial has the general form:

$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k = f(a) + f'(a)(k-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^k + \ldots$. Given the derivative and initial conditions in the problem, the third degree Taylor polynomial for y would be: $T_3(x) = 3 + 6(x - 2) + \frac{2}{2!}(x - 2)^2 + \frac{4}{3!}(x - 2)^3$. At $x = 3$, y would then equal 10.667.

2. Let R be the region bound by $y_1 = 2x^3 - 4x^2 - 8$ and $y_2 = 2x^2 + 8x - 8$.

(a) Find the area of R.

(a) First, set the two equations equal to each other to determine the bounds of R. The two curves intersect at $x = -1$, $x = 0$ and $x = 4$. Notice, that the "top" curve switches at $x = 0$, so be sure to write two separate integrals and add them together to determine the area: $\int_{-1}^{0}(2x^3 - 4x^2 - 8 - 2x^2 - 8x + 8)dx + \int_{0}^{4}(2x^2 + 8x - 8 - 2x^3 + 4x^2 + 8)dx = 62.5$

(b) Find the volume of the solid generated when R is revolved about the x-axis for

(b) Since the two curves and the axis of rotation are in the same form "$y =$", the washer method is the best way to solve for the volume of the solid: $\pi\int_{0}^{4}((2x^2 + 8x - 8)^2 - (2x^3 - 4x^2 - 8)^2)dx = 1492.11$.

(c) Find the volume of the solid generated when R is revolved about the line $x = 2$ when $-1 \le x \le 0$.

(c) Now, because the two curves and the axis of rotation are in different forms "$y =$" and "$x =$", cylindrical shells is the best way to solve for the volume. Notice the axis of rotation is not at $x = 2$, so the "radius" of the cylinder must be adjusted. Remember that when adjusting for the axis of rotation, always subtract the less positive from the more positive. In the original equation x is the radius of the cylinder and represents $x = 0$. $x = 2$ is more positive, so the new radius is $2 - x$. The formula is $2\pi\int_{-1}^{0}(2 - x)(2x^3 - 4x^2 - 8 - 2x^2 - 8x + 8)dx = \frac{113\pi}{15}$.

3. Consider the graph of the polar curve $r = 1 + 2\sin\theta$ for $0 \le \theta \le 2\pi$. Let S be the region bound between the inner and outer loops.

(a) Write an integral expression for the area of S.

(a) First determine the limits of the two loops. The inner loop spans $0 \le \theta \le \pi$ and the outer loop spans $\pi \le \theta \le 2\pi$. The area under a single polar curve is found from the equation $A = \int_a^b \frac{1}{2}r^2 d\theta$. To determine the area of S, you must take the area under the outer loop and subtract the area under the inner loop $S = \frac{1}{2}\left(\int_0^{\pi}(1+2\sin\theta)^2 d\theta - \int_{\pi}^{2\pi}(1+2\sin\theta)^2 d\theta\right)$.

(b) Write expressions for $\frac{dx}{d\theta}$ and $\frac{dy}{d\theta}$ in terms of θ.

(b) You will need to convert between polar and Cartesian coordinates using the equations $x = r\cos\theta$ and $y = r\sin\theta$. Also, the derivative in polar coordinates of the curve is $\frac{dr}{d\theta} = 2\cos\theta$. Use the product rule to integrate the equations for x and y in terms of θ: $\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta$ and $\frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta$. Plug in the value of $\frac{dr}{d\theta}$ into both equations to get $\frac{dx}{d\theta} = 2\cos_2\theta - \sin\theta - 2\sin^2\theta$ and $\frac{dy}{d\theta} = \cos\theta(1 + 4\sin\theta)$.

(c) Write an equation in terms of x and y for the line normal to the graph of the polar curve at the point where $\theta = \frac{3\pi}{2}$. Show your work.

(c) At $\theta = \frac{3\pi}{2}$, $x = 0$ and $y = 1$. The slope of the tangent $\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{\cos\theta(1+4\sin\theta)}{2-\sin\theta}$. At $\theta = \frac{3\pi}{2}$, $\frac{dy}{dx} = 0$.

Thus, the slope of the normal line is undefined, so the normal is parallel to the y-axis and the equation is $y = 1$.

4. For $t \geq 0$, a particle is moving along a curve so that its position at time t is $(x(t), y(t))$. At time $t = 3$, the particle is at position (3,1). It is known that $\frac{dx}{dt} = e^{-2t}(t + 1)^2$ and $\frac{dy}{dt} = \cos^2 t$.

(a) Is the horizontal movement to the left or to the right at time $t = 3$? Find the slope of the particle's path at $t = 3$.

(a) If $\frac{dx}{dt} < 0$, the particle is moving to the left. If $\frac{dx}{dt} > 0$, the particle is moving to the right. At $t = 3$, $\frac{dx}{dt} = e^{(-2)(3)}(1 + 3)^2 = \frac{16}{e^6}$, so $\frac{dx}{dt} > 0$. The particle is moving to the right.

The slope of the particle's path is $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$. So, at $t = 3$, $\frac{dy}{dx} = \frac{\cos^2 3}{e^{(-2)(3)}(1+3)^2} \approx 43.9327$.

(b) Find the y-coordinate of the particle's position at time $t = 6$.

(b) We know the y-coordinate at $t = 3$ is 1, so to find the y-coordinate at $t = 6$, we must integrate from $t = 3$ to $t = 6$ and add that to the y-coordinate at $t = 3$: $y(6) = 1 + \int_3^6 \cos^2 t\, dt \approx 2.43571$.

(c) Find the speed and acceleration of the particle at $t = 6$.

(c) $\frac{dx}{dt}$ and $\frac{dy}{dt}$ represent the components of the velocity vector with respect to time. To determine the speed, we need to find the magnitude of the velocity vector: $s = \sqrt{(x'(t))^2 + +(y'(t))^2}$. At $t = 6$, the speed would be: $s = \sqrt{\left(3^{(-2)(6)}(1+6)^2\right)^2 + \left(\cos^2 6\right)^2} \approx 0.967822$.

The acceleration, however, is the first derivative of the velocity vector, so it will be a vector, too: $(x''(t), y''(t))$. For this problem, the acceleration vector is: $(-2t(t + 1)e^{-2t}, -2t\sin t\cos t)$. At $t = 6$, the acceleration vector is $(-2(6)(6 + 1)e^{-2(6)}, -2(6)\sin 6\cos 6)$, so $(-5.16 \times 10^{-4}, 0.536573)$.

(d) Find the distance traveled by the particle from time $t = 3$ to $t = 6$.

(d) The distance traveled will be the length of the curve the particle travels, so the distance is found by evaluating: $D = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$. For this problem, the integral will be set up like this: $D = \int_3^6 \sqrt{\left(e^{-2t}(t+1)^2\right)^2 + \left(\cos^2 t\right)^2}\, dt$. Use your calculator to evaluate this integral and the distance is 1.43634.

5. A particle's position in the xy-plane at any time t is given by $x = 3t^3 - 4$ and $y = 2t^5 - 3t^3$. Find:

(a) The x and y components of the particle's velocity.

(a) The velocity is just the first derivative of the position functions, so the x and y components are: $\frac{dx}{dt} = 9t^2$ and $\frac{dy}{dt} = 10t^4 - 9t^2$ or $(9t^2, 10t^4 - 9t^2)$.

(b) $\frac{dy}{dt}$ at $t = 3$.

(b) $\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{10t^4 - 9t^2}{9t^2} = \frac{10}{9}t^2 - 1$. At $t = 3$, $\frac{dy}{dx} = \frac{10}{9}(3)^2 - 1 = 9$.

(c) The acceleration of the particle at $t = 3$.

(c) The acceleration is the second derivative of the position function or the first derivative of the velocity function. $\frac{d^2x}{dt^2} = 18t$ and $\frac{d^2y}{dt^2} = 40t^3 - 18t$. Since we are asked for the acceleration of the particle, not the components, we must find $\frac{d^2y}{dx^2} = \frac{\frac{d^2y}{dt^2}}{\frac{d^2x}{dt^2}} = \frac{40t^3 - 18t}{18t} = \frac{20}{9}t^2 - 1$. At $t = 3$, the acceleration is then

$\frac{d^2y}{dx^2} = \frac{20}{9}3^2 - 1 = 19$.

(d) The time(s) when the particle is changing direction.

(d) The particle will change direction when the velocity is zero, but the acceleration is not. Therefore, set the velocity function equal to zero and solve for t, and see if the acceleration is zero or not at that time. $\frac{dy}{dx} = \frac{10}{9}t^2 - 1 = 0$ when $t = \pm\frac{3\sqrt{10}}{10}$. Since we are only dealing with positive values of time, ignore the negative solution when checking the acceleration. $\frac{d^2y}{dx^2} = \frac{20}{9}\left(\frac{3\sqrt{10}}{10}\right)^2 - 1 = 1$. Since the velocity is zero and the acceleration is 1 at time $t = \frac{3\sqrt{10}}{10}$, the particle is chaning direction at that time.

6. Let $y_1' = \frac{3x^2}{y_1^2}$ and $y_2' = 2x^3 y_2 - xy_2$.

(a) If $x = 0$ and $y = 6$, find y_1.

(a) To find y_1, separate the variables in the derivative and plug in the given point to solve for the equation.

$\frac{dy_1}{dx} = \frac{3x^2}{y_1^2}$ becomes $y_1^2\, dy = 3x^2dx$, so $y_1^3 = 3x^3 = C$

With the initial condition, $y_1^3 = 3x^3 + 216$.

(b) If $x = -2$ and $y = e^2$, find y_2.

(b) Repeat the same process in part (a) for the second equation and the new initial condition. The final solution will be $\ln|y_2| = \frac{x^4}{2} - \frac{x^2}{2} - 4$.

(c) Use Euler's method to approximate y_1 when $x = 3$. Start at $x = 0$ using three steps. Check your answer against the real value of y_1 at $x = 3$. Is this a reasonable approximation?

(c) Using Euler's method, $y_n = y_{n-1} + h(y'_{n-1})$. At $x = 0$, $y_1 = 6$, $y_1' = 0$; at $x = 1$, $y_1 = 6$, $y_1' = \frac{1}{12}$; at $x = 2$, $y_1 = \frac{73}{12}$, $y_1' = \frac{1728}{5329}$; and at $x = 3$, $y_1 = 6.4076$.

The actual value for $y_1(3) = 6.67194$. The difference is 3.96%, so this is a reasonable approximation.